# Pomeron with a running coupling constant: intercept and slope

M. A. Braun [*]

Department of Particle Physics, University of Santiago de Compostela,

15706 Santiago de Compostela, Spain

October 1994

**Abstract** The equation for two reggeized gluons with a running QCD coupling constant, proposed earlier on the basis of the bootstrap condition, is investigated by the variational technique in the vacuum channel. The pomeron intercept and slope are calculated as a function of the infrared cutoff parameter $m$ ("the gluon mass"). The intercept depends weakly on $m$ staying in the region $0.3 - -0.5$ for values of $m$ in the range $0.3 - -1.0 \ GeV$. The slope goes down from $5.0 \ GeV^{-2}$ to $0.08 \ GeV^{-2}$ in this range. The optimal value of $m$ seems to lie in the region $0.6 - -0.8 \ GeV$ with the intercept $0.35$ and slope $0.35 - -0.15$. The character of the pomeron singularity: pole or cut, is also discussed.

**US-FT/17-94**

---

[*]On leave of absence from the Department of high-energy physics, University of S. Petersburg, 198904 S. Petersburg, Russia

**1. Introduction.** We have recently proposed a new method to incorporate a running coupling constant into the equation governing the behaviour of two reggeized gluons [ 1 ]. It is based on the so-called "bootstrap condition" of L.Lipatov [ 2 ], which requires that the solution in the gluon colour channel should be the reggeized gluon itself. The bootstrap condition leads to a relation between the gluon interaction $U$ and its Regge trajectory $\omega$: both have to be expressed via the same function $\eta$, so that in the momentum space

$$\omega(q) = -(3/2)\eta(q) \int (d^2q_1/(2\pi)^2)/\eta(q_1)\eta(q_2) \tag{1}$$

and

$$U(q, q_1, q_1') = -3(\eta(q_1)/\eta(q_1') + \eta(q_2)/\eta(q_2'))/\eta(q_1 - q_1') + 3\eta(q)/\eta(q_1')\eta(q_2') \tag{2}$$

In (2) $q$ is the total momentum of the two interacting gluons. In both equations $q_1 + q_2 = q$.

The asymptotical behaviour of $\eta(q)$ at high momenta can be found from the comparison of the pomeron equation at $q = 0$ and the standard Gribov-Lipatov-Altarelli-Parisi evolution equation in the double logarithmic approximation. With a running QCD coupling constant $g(q)$ we obtain at high $q$

$$1/\eta(q) = g^2(q)/2\pi q^2 = 2\pi/bq^2 \ln q^2/\Lambda^2 \tag{3}$$

where

$$b = (11 - (2/3)N_F)/4 \tag{4}$$

and $\Lambda \simeq 200 \; Mev$ is the standard QCD parameter. With a fixed coupling constant (the BFKL pomeron [ 3 ]) $1/\eta(q)$ results proprtional to $q^2$. It follows that the introduction of a running coupling constant changes the pomeron equation already at leading order, so that it cannot be studied by a perturbative approach like in [ 4 ].

The behaviour of $\eta(q)$ at low $q^2$ remains unknown and, as we want to stress, hardly amenable to any theoretical approach, since it belongs to the nonperturbative confinement region. More than that, the two-gluon equation itself looses sense at low $q \sim \Lambda$, where gluons cease to exist as individual particles. Therefore the behaviour of $\eta(q)$ at low $q^2$ plays the role of a boundary condition to be supplemented to the pomeron equation, which effectively takes into account the confinement effects. One can at most try to parametrize it in some simple manner and study the consequences. In [ 1 ] we have chosen a parametrization

$$1/\eta(q) = 2\pi/b(q^2 + m^2) \ln((q^2 + m^2)/\Lambda^2) \tag{5}$$

where the parameter $m \geq \Lambda$ has a meaning of an effective gluon mass.

The asymptotical behaviour of both the gluon trajectory and interaction with the choice (5) was studied in [ 1 ]. Taking for them simple analytic expressions which interpolate

between low and high values of their arguments, the pomeron intercept was also studied by the variational technique in [ 1 ].

In this note we investigate the pomeron intercept with exact values for the trajectory and interaction, which follow from (1) and (2) with the choice (5). We confirm that the intercept depends very weakly on the value of the gluon mass $m$, as was found in [ 1 ] for approximate $\omega$ and $U$. Its absolute values turn out, however, somewhat higher: for $m$ growing from 0.3 $GeV$ to 2 $GeV$ the intercept falls from 0.5 to 0.25.

Since we know the pomeron equation with a running coupling constant also for $q \neq 0$, we are in a position to calculate the slope of the pomeron trajectory at $q = 0$. It happily results positive, its values depending very strongly on $m$: in the same interval $m = 0.3 - -2.\ GeV$ the slope goes down from 5.0 $GeV^{-2}$ to 0.01 $GeV^{-2}$. From the experimental observations then the optimal value of $m$ lies in the region of the $\rho$-meson mass $m = 0.6 - -0.8\ GeV$.

We finally discuss the character of the pomeron singularity in the $j$-plane. As it seems, it depends on the chosen behaviour of $\eta(q)$ at $q \to 0$. For the choice (5) it turns out to be a moving pole, although we cannot exclude that other choices may result in a moving cut.

**2. The intercept.** For the forward scattering, $q = 0$, the pomeron equation aquires the form of a Shroedinger equation in two dimensions

$$H\psi = E\psi \tag{6}$$

where the Hamiltonian is a sum

$$H = T + U + Q \tag{7}$$

of a kinetic energy

$$T(q) = -2\omega(q) \tag{8}$$

a local interaction $U(r)$ with a Fourier transform

$$U(q) = -6/\eta(q) \tag{9}$$

and a separable interaction

$$Q = (1/12)\eta(0)|U\rangle\langle U| \tag{10}$$

The intercept $\Delta$ is the ground state energy of (6) with a minus sign.

In [ 1 ] the analytic forms were used for $T(q)$ and $U(r)$ which interpolate between small and large values of their respective arguments

$$T(q) = T(0) + \ln\ln(q^2 + m^2), \quad U(r) = \ln\ln(1/r^2 + m^2) \tag{11}$$

(in units $3/b$, $\Lambda = 1$). Numerical calculations with $\eta(q)$ given by (5) lead to $T(q)$ and $U(r)$ shown in Table 1. for $m = 3.0$ in the regions where they noticeably differ from their

approximate expressions (11) (shown by dashed curves). As one observes, the largest error in using (11) is for $U(r)$ at large $r$, where the exact potential falls exponentially, whereas an approximate one only as $1/r^2$. Since the potential is attractive, one might think that with the approximate potential the ground state level should lie deeper. However the approximate kinetic energy lies systematically above the exact one. This effect turns out to be dominating.

Taking exact numerical values for $T(q)$ and $U(r)$ we investigated the ground state energy $E_0$ by the standard variational method. As in [ 1 ] trial functions were chosen as linear combinations of the two-dimensional harmonic oscillator functions for zero angular momentum

$$\psi(r^2) = \sum_{k=0}^{N} c_k L_k(ar^2) \exp(-ar^2/2) \tag{12}$$

with $L_k$ the Lagerre polinomials and $a$ and $c_k$ variational parameters. Up to 16 polinomials have been included in the calculations. The resulting intercept is shown in Table 2. (the second column) as a function of $m$ for $\Lambda = 0.2 \ GeV$ and three flavours ($b = 9/4$). As one observes, it depends on $m$ rather weakly except for the values of $m$ quite close to $\Lambda$. For reasonable values of $m$ in the range $0.3 - -1. \ GeV$ it goes down from 0.5 to 0.3. Comparing these values with the ones found in [ 1 ] with (11) we find that the exact trajectory and interaction lead to the intercept twice as high.

**3. The slope.** To determine the slope at $q = 0$ we can use the perturbative approach. For small values of $q$ we present the Hamiltonian in the form

$$H = H_0 + W(q) \tag{13}$$

where $H_0$ is the Hamiltonian for $q = 0$ and $W$ is the perturbation proportional to $q^2$. The slope $\alpha'$ is then calculated as the average of $W$ in the ground state $\psi_0$ at $q = 0$ divided by $q^2$.

At $q \neq 0$ the pomeron equation retains its form (6). The components of the Hamiltonian (7) are now given by

$$T = -\omega_1 - \omega_2 \tag{14}$$

$$U = -3(\sqrt{\eta_1/\eta_2} V \sqrt{\eta_2/\eta_1} + \sqrt{\eta_2/\eta_1} V \sqrt{\eta_1/\eta_2}) \tag{15}$$

$$Q = 3\eta(q)|1/\sqrt{\eta_1\eta_2}\rangle\langle 1/\sqrt{\eta_1'\eta_2'}| \tag{16}$$

Here and in the following $\omega_1 = \omega(q_1)$, $\eta_1 = \eta(q_1)$ and so on.

Let $q_1 = (1/2)q + l$, $q_2 = (1/2)q - l$. To find $W(q)$ we develop (14)-(16) in powers of $q$ up to the second order. Straightforward calculations lead to

$$T = T_0 - (1/8)[qr[qr, T_0]] \tag{17}$$

$$U = U_0 + (1/2)[(ql)\zeta_1[(ql)\zeta_1, U_0]] \tag{18}$$

$$Q = Q_0 + q^2\zeta_3 Q_0 - \{(1/4)q^2\zeta_1 + (1/2)(ql)^2(\zeta_2 - \zeta_1^2), Q_0\} \tag{19}$$

where $T_0, U_0$ and $Q_0$ correspond to $q = 0$ and are given by (8)-(10). We have denoted

$$\zeta_1(l) = (\ln\mu + 1)/\mu\ln\mu, \quad \zeta_2(l) = 1/\mu^2\ln\mu, \quad \zeta_3 = \zeta_1(0)$$

where $\mu = m^2 + l^2$. Note that in the double commutators (17) and (18) one can change $T_0$ and $U_0$ to $T_0 + U_0 = H_0 - Q_0$, since the added terms commute with the respective operators. In this manner we finally obtain

$$W = -(1/8)[qr[qr, H_0 - Q_0]] + (1/2)[ql\zeta_1[ql\zeta_1, H_0 - Q_0]]$$

$$+ q^2\zeta_3 Q_0 - \{(1/4)q^2\zeta_1 + (1/2)(ql)^2(\zeta_2 - \zeta_1^2), Q_0\} \tag{20}$$

Taking the average of (20) in the ground state $\psi_0$ we present the corresponding change in energy $\Delta E$ as a sum of three terms

$$\Delta E = \sum_{n=1}^{3} \Delta E^{(n)} \tag{21}$$

Here terms with $n = 1, 2$ and 3 correspond to parts of (20) proportional to the ground state energy $E_0$, containing $T_0 + U_0$ and containing $Q_0$ respectively.

$$\Delta E^{(1)} = E_0\langle(-1/4)(qr)^2 + (ql)^2\zeta_1^2\rangle \tag{22}$$

$$\Delta E^{(2)} = \langle(1/4)(qr)(T_0 + U_0)(qr) - (ql)\zeta_1(T_0 + U_0)(ql)\zeta_1\rangle \tag{23}$$

and

$$\Delta E^{(3)} = 3\eta_0\langle\psi_0|((1/4)(qr)^2 + q^2(\zeta_3 - (1/2)\zeta_1) - (ql)^2\zeta_2)V\rangle\langle V|\psi_0\rangle \tag{24}$$

where we denoted $\langle\psi_0|A|\psi_0\rangle \equiv \langle A\rangle$.

In most of these averages one can immediately integrate over the azimuthal angle substituting $(ql)^2 \to (1/2)q^2l^2$ and $(qr)^2 \to (1/2)q^2r^2$. Two averages, however, are a bit more complicated.

$$A_1 \equiv \langle(qr)T_0(qr)\rangle, \quad A_2 \equiv \langle(ql)\zeta_1 U_0(ql)\zeta_1\rangle$$

The first one is evidently a momentum space average in the state $(qr)\psi_0(l^2) = 2i(ql)\psi_0'(l^2)$, so that $A_1 = 2q^2\langle\psi_0'|l^2T_0|\psi_0'\rangle$. The second average, also in the momentum space, reduces to that of a nonlocal potential $U_1$: $A_2 = (1/2)q^2\langle U_1\rangle$ where $U_1$ possesses a kernel

$$U_1(l_1, l_2) = -6\zeta_1(l_1)\zeta_1(l_2)(l_1 l_2)/\eta(l_1 - l_2) \tag{25}$$

The final formula for the slope consists of three terms in correspondence with (22)-(24)

$$\alpha' = \sum_{n=1}^{3} \alpha_n' \tag{26}$$

They are

$$\alpha'_1 = E_0 \langle (-1/8)r^2 + (1/2)l^2\zeta_1^2 \rangle \tag{27}$$

$$\alpha'_2 = \langle (1/8)r^2 U_0 - (1/2)l^2\zeta_1^2 T_0 - (1/2)U_1 \rangle + (1/2)\langle \psi'_0|l^2 T_0|\psi'_0 \rangle \tag{28}$$

$$\alpha'_3 = 3\eta_0 \langle \psi_0|((1/8)r^2 + \zeta_3 - (1/2)\zeta_1 - (1/2)l^2\zeta_2)V \rangle \langle V|\psi_0 \rangle \tag{29}$$

We have calculated $\alpha'$ according to these formulas taking for the ground state $\psi_0$ the trial function (12) with the parameters which minimize the energy. The results are shown in Table 2. (the third column). They strongly depend on $m$, reaching quite large values for $m$ close to $\Lambda$ and rapidly falling with its growth to values of the order 0.01 $GeV^{-2}$ for $m = 10\lambda$.

**4. Pole or cut?** We finally want to comment on the character of the pomeronic singularity in the $j$ plane for our eqation. It is easy to show that with the choice of $\eta(q)$ according to (5) it is a moving pole. To prove it we have to demonstrate that the ground state wave function $\psi_0$ is normalizable.

With $\eta(q)$ given by (5), the interaction falls exponentially for large intergluon distances $r \to \infty$, so that the equation reduces to

$$(T(q) - E)\psi(r) = 0 \tag{30}$$

where $q^2 = -\Delta$. Evidently the asymptotics of the solution is a linear combination of zero order Bessel functions

$$\psi(r) \simeq a\mathrm{H}_0^{(1)}(qr) + b\mathrm{H}_0^{(2)}(qr) \tag{31}$$

where $q$ is determined from the equation

$$T(q) = E \tag{32}$$

However, for positive $q^2$ the kinetic energy $T(q)$ is greater than a positive threshold value $T(0)$ (Table. 1). Then for $E \geq T(0)$ the asymptotics (31) is oscillating and $\psi$ is not normalizable. For $E < T(0)$, and for $E < 0$ in particular, Eq. (32) gives pairs of complex conjugate solutions $q = q_1 \pm q_2$ with $q_2 > 0$. Then evidently the function $\psi(r)$ becomes exponentially falling at large $r$ like $\exp(-q_2 r)$

To show its normilizability we have additionally to study its behaviour at $r = 0$, that is, at large momenta $q$. With the asymptotics of the kinetic energy clear from (11), the equation at $q \to \infty$ becomes

$$\ln\ln q^2\psi(q) + bm^2\ln m^2/q^2\ln q^2 = (1/\pi)\int d^2q'\psi(q')/((q-q')^2+m^2)\ln((q-q')^2+m^2) \tag{33}$$

Two alternatives are possible: either

$$\psi(r=0) = \int (d^2q/2\pi)\psi(q) < \infty \tag{34}$$

or $\psi(r = 0) = \infty$. We want to show that the first one is realized. In fact, assume that $\psi(r = 0) = \infty$. Introduce a function $f(\xi) = q^2 \ln q^2 \psi(q)$ with $\xi = \ln \ln q^2$. By assumption

$$\int d\xi \, f(\xi) = \infty$$

After calculating the asymptotical behaviour of the integral term in (33) one arrives at an equation for $f$ of the form

$$\xi f(\xi) = \int^{\xi} d\xi' (f(\xi') + f(\xi))$$

which can never be satisfied, since the second term on the righthand side exactly cancels the lefthand side term. Thus (34) is true. Then the integral term in (33) behaves like $1/q^2 \ln q^2$ at high $q$, which contribution should be cancelled by the one from the separable interaction (the second term on the lefthand side). The first term, coming from the kinetic energy, then represents a subdominant correction. With (34) fulfilled, the function $\psi(r)$ is well-behaved for all $r$ and is normalizable. Then solutions for $E_0 < 0$ should belong to a discrete spectrum.

We have to point out, however, that this conclusion depends heavily on our choice of $\eta(q)$, which assumes that it is different from zero at $q = 0$. As discussed in the beginning, actually we do not know anything about the behaviour of $\eta(q)$ at $q << \Lambda$ nor does it make any physical sense. More general parametrizations, like

$$\eta(q) = (b/2\pi)(q^2 + m_1^2) \ln(q^2 + m^2)$$

with two different masses $m$ and $m_1$ are equally admissible. In particular we can take $m_1 = 0$ and choose

$$\eta(q) = (b/2\pi)q^2 \ln(q^2 + m^2) \qquad (35)$$

when $\eta(0) = 0$. This choice seems to have some advantages: now the trajectory will go through 1 at $q = 0$ ( for the "physical" gluon) and also the solution will satisfy the condition $\psi(q = 0) = 0$, which follows from the colour current conservation if one goes down to $q = 0$. However these advantages are completely spurious: with confinement one can never go as far as $q = 0$ and should always stay at momenta $q > \Lambda$ in coloured channels. So, in our opinion, the choice (35) is nothing better than our earlier (5).

It however leads to a considerably more complicated gluonic Hamiltonian. With (35) the kinetic energy and interaction do not exist separately, but have to be taken together in the integral kernel exactly in the same manner as with the BFKL equation [ 3 ]. The infrared behavior of this new Hamiltonian is completely different from the one with (5) and becomes similar to that of the BFKL Hamiltonian. With that, the equation with a running coupling constant does not possess the scaling symmetry and hardly admits an analytic study. We cannot say anything definite about the pomeron singularity for it: it might well be a cut as for the BFKL equation (only a moving cut, contrary to the BFKL case).

Thus we conclude that in the real world with a running coupling constant the equation for the pomeron is sensitive to the confinement effects, that is, to the behaviour of the trajectory and interaction in the infrared region. The dependence on this behaviour is minimal (and quite weak, indeed) for the intercept, whose values, after all, result not very different from those extracted from the BFKL pomeron for reasonable values of the coupling constant. However the infrared behavior has a strong influence on the slope value and may be essential for the character of the pomeron singularity.

**References.**

1. M.Braun, Univ. of Santiago preprint US-FT/11-94

2. L.Lipatov in "Petrturbative QCD", Ed. A.Mueller, Advanced Series in High Energy Physics (World Scientific, Singapore, 1989)

3. E.Kuraev, L.Lipatov and V.Fadin, Sov. Phys. JETP, **45** (1977) 199
Ya.Balitzky and L.Lipatov, Sov. Phys. J.Nucl. Phys.,**28** (1978)822

4. V.Fadin and L.Lipatov, Nucl. Phys. **B406** (1993) 259

## Table 1.  Exact and approximate gluon trajectory and potential

| $q^2$ | $T$ | $T_{approx}$ | $r^2$ | $-U$ | $-U_{approx}$ |
|---|---|---|---|---|---|
| 0.37 | 0.133 | 0.138 | 0.0183 | 1.319 | 1.293 |
| 1.00 | 0.147 | 0.178 | 0.0498 | 1.044 | 1.008 |
| 2.72 | 0.241 | 0.372 | 0.135 | 0.695 | 0.637 |
| 7.39 | 0.593 | 0.767 | 0.368 | 0.302 | 0.241 |
| 20.1 | 1.013 | 1.139 | 1.00 | 0.0426 | 0.0468 |
| 54.6 | 1.293 | 1.424 | 2.72 | 2.78E-4 | 6.77E-3 |
| 148. | 1.515 | 1.646 | 7.39 | 4.03E-10 | 9.25E-4 |
| 403. | 1.698 | 1.829 | | | |
| 1097 | 1.852 | 1.983 | | | |
| 2981 | 1.985 | 2.116 | | | |
| 8103 | 2.103 | 2.234 | | | |

## Table 1. captions

The first column gives values of the gluon momentum squared in units $\Lambda^2$. In the second and third columns the corresponding exact and approximate (Eq. (11)) values of the gluon trajectory are presented in units $6/b$. The fourth column gives the values of the inter-gluon distance squared in units $\Lambda^{-2}$. In the last two columns the corresponding exact and approximate values of the gluon potential (with minus sign) are given (in units $3/b$)

# Table. Pomeron intercept and slope

| $m$ | $\Delta$ | $\alpha'$ |
|---|---|---|
| 0.210 | 1.02 | 86.0 |
| 0.245 | 0.687 | 13.8 |
| 0.283 | 0.579 | 5.77 |
| 0.4 | 0.451 | 1.32 |
| 0.6 | 0.372 | 0.346 |
| 0.8 | 0.336 | 0.150 |
| 1.0 | 0.311 | 0.0812 |
| 1.41 | 0.280 | 0.0333 |
| 2.0 | 0.256 | 0.0138 |

# Table 2. captions

The first column gives values of the gluon mass in $GeV$ In the second column the corresponding pomeron intercepts are presented. The third column gives the pomeron slopes in $GeV^{-2}$. The QCD parameter $\Lambda$ has been taken to be 0.2 $GeV$ for three flavours ($b = 9/4$).